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ON THE THEORY OF STABILITY ON A GIVEN TIME INTERVAL

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A problem of stability on an infinite time interval is considered using the formulation of [1] which can also be used on a finite time interval. Certain comparisons with the formulation due to Liapunov are made. A sufficient condition for the asymptotic stability as well as the necessary and sufficient condition of stability for a linear homogeneous autonomous system, and sufficient conditions of asymptotic stability and instability for a nonlinear autonomous system are obtained.

1. We consider a problem of stability using the following formulation [1]:

Definition 1.1. An unperturbed process is called stable if a matrix $G(t)$ exists in a given class K_Δ^ω such that for a sufficiently small $\rho > 0$ any perturbation of $x(t)$ of the process, the initial value $x(t_0) = x_0$ of which satisfies the condition

$$(G^{-1}(t_0)\,x_0,\ G^{-1}(t_0)\,x_0) \leqslant \rho^2 \tag{1.1}$$

and for all $t > t_0$ satisfies the condition

$$(G^{-1}(t)\,x(t),\ G^{-1}(t)\,x(t)) \leqslant \rho^2 \tag{1.2}$$

Under the class K_Δ^ω we understand the collection of $n \times n$ matrices $G(t) = (G_1(t), \ldots, G_n(t))$ over the field of complex numbers satisfying, on $[t_0, \infty)$ the conditions that $|\det G(t)| \neq 0$ and, that the Hermitian norm of the columns $G_1(t), \ldots, G_n(t)$ coincides with a given positive function $\omega(t)$, i.e. that $\|G_j(t)\| = \omega(t)$ $(j = 1, 2, \ldots, n)$.

Definition 1.2. An unperturbed process is called asymptotically stable on $[a, \infty)$, if it is stable on $[a, \infty)$ and for any $t_0 \in [a, \infty)$ a value $\rho = \rho(t_0) > 0$ exists such that all perturbations $x(t)$ of the process satisfying the condition (1.1) have the property stating that $\lim \|x(t)\| = 0$ as $t \to \infty$.

Using this formulation we give certain conditions of stability on an unlimited interval and of asymptotic stability of motion.

2. Let us consider the equation

$$dx / dt = X(t, x) \tag{2.1}$$
$$X(t, x) \in C_{t,x}^{(0,1)}(Z), \quad Z = I \times D, \quad I = \{a < t < \infty\}$$

Here D denotes an open region belonging, in general, to the n-dimensional complex vector space R^n, while X and x are column matrices of the type $n \times 1$, with $X(t, 0) \equiv 0$.

We note that for a bounded function $\omega(t)$: a) the system is unstable under the given formulation if at least one solution unbounded on I exists originating in a sufficiently small neighbourhood of the coordinate origin and, b) if the system is stable, then every solution of this system originating in a sufficiently small neighbourhood of the coordinate origin is bounded on I.

The example which follows illustrates the fact that the boundedness of the solutions of the system does not represent a sufficient criterion for its stability.

Consider the equation

$$\frac{dx}{dt} = -\frac{t-b}{\sigma^2}x, \quad \sigma = \text{const} > 0$$

the trivial solution ($x \equiv 0$) of which is Liapunov stable. The nontrivial solution has the form

$$x(t) = \frac{c}{\sigma}E(t), \quad E(t) = \exp\left[-\frac{(t-b)^2}{2\sigma^2}\right]; \quad x(t_0) = \frac{c}{\sigma}E(t_0) \quad (t_0 < b)$$

In the case of a scalar differential equation, the ρ_ω tubes have the form

$$V(t, x) \equiv \omega^{-2}(t)\,|\,x\,|^2 \leqslant \rho^2$$

When $|\,c\,| \leqslant \sigma\rho E^{-1}(t_0)$ and $\omega(t_0) = 1$ we obtain $V(t_0, x(t_0)) = |\,x(t_0)\,|^2 \leqslant \rho^2$, and e.g.

$$V(b, x(b)) = \frac{1}{\omega^2(b)}\rho^2 \exp\frac{(t_0-b)^2}{\sigma^2}$$

This shows that depending on $\omega^2(b)$ the quantity $V(t, x(t))$ can be greater than ρ^2 near the point $t = b$, although $x(t)$ is bounded in the interval $[t_0, \infty)$.

This example shows that a Liapunov-stable system may lack the stability as defined by Definition 1.1.

We shall show that the stability under the given formulation always implies the Liapunov stability. Let $\omega(t) \leqslant \omega_0$ and the system (2.1) be stable in the sense of Definition 1.1. Then every solution of this system satisfying the condition (1.1) will also satisfy the condition (1.2). From (1.2) by virtue of inequality $\lambda_i(H(t)) \geqslant (\sqrt{2}\omega_0)^{-1}$ $(t_0 \leqslant t < \infty)$ (see [2]) where $H(t) = G^{-1*}(t)\,G^{-1}(t)$ for all $t > t_0$, we have $\|\,x(t)\,\| < (\sqrt{2}\omega_0)^{1/2}\rho$. Let ε be an arbitrarily small positive number. When $\rho < \varepsilon(\sqrt{2}\omega_0)^{-1/2}$ we have $\|\,x(t)\,\| < \varepsilon$ for all solutions the initial values of which satisfy the condition $\|\,x(t_0)\,\| < \sqrt{\varepsilon} = \delta$. This implies that the system is Liapunov-stable.

In what follows, the given positive function $\omega(t)$ will be treated as a constant.

3. Let us consider the system

$$dx\,/\,dt = Ax \tag{3.1}$$

where A is a constant $n \times n$ matrix.

Theorem 3.1. The system (3.1) is asymptotically stable if all characteristic roots λ_j $(j = 1, \ldots, n)$ of the matrix A have negative real parts.

Proof. Let $\operatorname{Re}\lambda_j < 0$ $(j = 1, \ldots, n)$. Then for any positive definite Hermitian matrix $W = \text{const}$ a positive definite Hermitian matrix $H = \text{const}$ exists such that

$$A^*H + HA = -2W \tag{3.2}$$

A solution of (3.2) for H has the form

$$H = 2\int_0^\infty e^{A^*\tau} W e^{A\tau}\, d\tau$$

for any W (see e.g. [3]). Let us choose W such that the following relation holds for H:

$$\frac{1}{n}\sum_{i=1}^{n} \mu_i^{-1} = \omega^2$$

where μ_i $(i = 1, \ldots, n)$ are the eigenvalues of the Hermitian matrix H. Consequently, using a lemma given in [4], we can write the matrix H in the form $H = K^{-1*}K^{-1}$ where the relation $\| K_j \| = \omega$ $(j = 1, \ldots, n)$ holds for the columns of the matrix $K = (K_1, \ldots, K_n)$.

The function $V(x) = (Hx, x)$ defines the ρ_ω - tube

$$V(x) \equiv (K^{-1}x, K^{-1}x) = \rho^2$$

since $K \in K_\Delta{}^\omega$. Its derivative with respect to t can be written, by virtue of the system (3.1), in the form $V^\cdot(x) = -2(Wx, x)$. When $x(t) \not\equiv 0$ we have $V^\cdot(x(t)) < 0$ on $[a, \infty)$ and this implies that $V(x(t)) \leqslant V(x(a))$ for all $t > a$. From the last inequality follows the stability of (3.1).

On the other hand, since every solution of (3.1) has the form

$$x(t) = \sum_{j=1}^{m} e^{\lambda_j t} P_i(t) \quad (m \leqslant n)$$

where $P_j(t)$ $(j = 1, \ldots, m)$ are polynomial matrices, we have

$$\lim_{t\to\infty} \| x(t) \| = 0$$

which completes the proof. The converse of the theorem is also true.

Theorem 3.2. The system (3.1) is stable if and only if all characteristic roots λ_j $(j = 1, \ldots, n)$ of the matrix A have nonpositive real parts, i.e. $\mathrm{Re}\,\lambda_j \leqslant 0$ $(j = 1, \ldots, n)$ where the equals sign occurs only if the corresponding elementary divisors are simple.

Proof. We assume that $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of the matrix A corresponding to various Jordan cells and, that the transformation $x = Ky$ reduces (3.1) to the form

$$dy / dt = I(\lambda) y, \quad I(\lambda) = \mathrm{diag}\,\{I_1(\lambda_1), \ldots, I_p(\lambda_p)\} \tag{3.3}$$

Necessity. Let the system (3.1) be stable. From the existence of an eigenvalue λ_s of the matrix A such that $\mathrm{Re}\,\lambda_s > 0$ or $\mathrm{Re}\,\lambda_s = 0$ and the corresponding elementary divisor is of order $K_s > 1$, it follows that the system (3.1) has an unbounded solution. Since this contradicts the condition of stability of the system (3.1), the necessity is proved.

Sufficiency. Let λ_j ($j = 1, \ldots, q$; $q < p$) denote all characteristic roots of the matrix A which have the corresponding different Jordan cells and $\operatorname{Re} \lambda_j < 0$, and let λ_s ($s = q + 1, \ldots, p$) denote all characteristic roots of A where $\operatorname{Re} \lambda_s$ 0 and all λ_s admit only the simple, elementary divisors. Then we can partition the column matrix $y(t)$ in (3.3) into two blocks in such a manner, that the number in the first block is equal to $r = k_1 + k_2 + \ldots + k_q$ (k_j denotes the order of the cell $I_j(\lambda_j)$). After this the system (3.3) can be separated into two independent subsystems

$$dy^{(1)}/dt = My^{(1)}, \quad dy^{(2)}/dt = \Lambda y^{(2)} \tag{3.4}$$
$$M = \operatorname{diag}\{I_1(\lambda_1), \ldots, I_q(\lambda_q)\}, \quad \Lambda = \operatorname{diag}\{I_{q+1}(\lambda_{q+1}), \ldots, I_p(\lambda_p)\}$$

All conditions of the theorem 3.1. hold for the first system of (3.4), therefore the system is stable i.e. a matrix $G_1(t) \in K_\Delta^{\omega}$ exists such that along the solution $y^{(1)}(t)$ the inequality

$$V_1(y^{(1)}(t_0)) \leqslant \rho_1^2 \tag{3.5}$$

for all $t > t_0$ implies that

$$V_1(y^{(1)}(t)) = (H_1 y^{(1)}, y^{(1)}) \leqslant \rho_1^2 \Big(\frac{1}{r} \sum_{i=1}^{r} \mu_i^{-1}(H_1) = \omega^2\Big), \quad H_1 = G_1^{-1*} G_1^{-1} \tag{3.6}$$

(($\rho_1 > 0$ is an arbitrary, sufficiently small number).

Let us consider the function (H_2 is a square matrix of the order $n - r$)

$$V_2(\eta(t)) \equiv (H_2 \eta(t), \eta(t)) = \omega^{-2} \| \eta(t) \|^2$$
$$H_2 = \operatorname{diag}\{\omega^{-2}, \ldots, \omega^{-2}\} \tag{3.7}$$

Its derivative with respect to t has, by virtue of the second system of (3.4), the form

$$V_0^{\bullet}(y^{(2)}(t)) = 2\omega^{-2} \sum_{k=}^{n-r} \operatorname{Re} \lambda_{q+k} \, | y_{q+k} |^2$$

This implies that $V_2^{\bullet}(y^{(2)}(t)) = 0$ on $[t_0, \infty)$. It means that when the initial value $y^{(2)}(t_0)$ is chosen so small that

$$V_2(y^{(2)}(t_0)) \leqslant \rho_2^2 \tag{3.8}$$

then we have

$$V_2(y^{(2)}(t)) \leqslant \rho_2^2 \tag{3.9}$$

($\rho_2 > 0$ is arbitrary and sufficiently small) for all $t > t_0$, i.e. the second system of (3.4) is also stable.

Any solution $y(t)$ of the system (3.3) has the form

$$y(t) = \begin{Vmatrix} y^{(1)}(t) \\ y^{(2)}(t) \end{Vmatrix} \tag{3.10}$$

where $y^{(1)}(t)$ and $y^{(2)}(t)$ represent the solutions of the first and second system of (3.4) respectively.

Let us take the function $V(y(t)) = (Hy(t), y(t))$, where $H = \text{diag}\{H_1 H_2\}$ and $n^{-1}[\mu_1^{-1}(H) + \ldots + \mu_n^{-1}(H)] = \omega^2$, defining the ρ_ω-tube: $V(y) \equiv (Hy, y) = \rho^2$. At the initial instant we have, by virtue of (3.5) and (3.8), the following relation along the solution (3.10):

$$V(y(t_0)) = V_1(y^{(1)}(t_0)) + V_2(y^{(2)}(t_0)) \leqslant \rho^2 \quad (\rho^2 = \rho_1^2 + \rho_2^2)$$

and for all $t > t_0$ we have, by virtue of (3.6) and (3.9),

$$V(y(t)) = V_1(y^{(1)}(t)) + V_2(y^{(2)}(t)) \leqslant \rho^2$$

(when ρ_1 and ρ_2 are chosen sufficiently small, $\rho > 0$ will also be sufficiently small). The theorem is proved.

4. Criteria of stability in the first approximation. Let the differential equation have the form

$$dx/dt = Ax + f(x), \quad f(x) = \text{col}(f_1(x), \ldots, f_n(x)) \tag{4.1}$$

where A is a constant $n \times n$ matrix and the functions $f_s(x) = f_s(x_1, \ldots, x_n)$ $(s = 1, \ldots, n)$ can be expanded in the region $\|x\| \leqslant L < \infty$ into series in powers of $x_1, \ldots, x_n$, the first terms of which are of at least second order. Below we shall also concern ourselves with the first order approximation to the system (4.1), namely

$$dx/dt = Ax \tag{4.2}$$

Theorem 4.1. The system (4.1) will be asymptotically stable if all characteristic roots of the matrix A have negative real parts.

Proof. Let $\text{Re}\,\lambda_j < 0$ $(j = 1, \ldots, n)$. Then by virtue of Theorem 3.1. the system (4.2) is asymptotically stable.

Consider the function

$$V(x) \equiv (Hx, x) \equiv (K^{-1}x, K^{-1}x)$$

constructed in Theorem 3.1. Its derivative with respect to t, by virtue of the system (4.1), has the form

$$V^{\cdot}(x(t)) = -2(Wx, x) + (Hx, f(x)) + (Hf(x), x) \tag{4.3}$$

Since the first terms in the expression $(Hx, f(x)) + (Hf(x), x)$ are of at least third order, the function (4.3) will, at sufficiently small x ($\|x\| \leqslant h < L$) be negative definite on $[a, \infty)$ for any functions $f_s(x)$ $(s = 1, \ldots, n)$. Consequently the inequality $V(x(t)) \leqslant V(x(t_0))$ holds for all $t > t_0 \geqslant a$, and this means that the system (4.1) is stable.

We shall show that

$$\lim_{t\to\infty} \| x(t) \| = 0 \tag{4.4}$$

is also true. Indeed, from (4.3) it follows that the function $V(x(t))$ is monotonously decreasing, therefore when $t\to\infty$, $V(x(t)) \to \nu \geqslant 0$ and we have, for all $t > t_0$,

$$V(x(t)) > \nu \tag{4.5}$$

We shall now prove that $\nu = 0$. Let $\nu \neq 0$. Since the function $V(x(t))$ is positive definite, $\nu > 0$. By virtue of the continuity of the function $V(x(t))$ from (4.5) follows $\| x(t) \| \geqslant \beta > 0$. However, since the form (4.3) is negative definite, we find that when $\| x(t) \| \geqslant \beta > 0$ the inequality $V^{\cdot}(x(t)) \leqslant -\gamma < 0$ holds on $[t_0, \infty)$. Consequently we have for all $t > t_0$

$$V(x(t)) \leqslant V(x(t_0)) - \gamma(t - t_0)$$

which is clearly impossible. Therefore $\lim_{t\to\infty} V(x(t)) = 0$ and that implies that (4.4) holds by virtue of the sign definiteness of the form $V(x(t))$. This completes the proof of the theorem.

Theorem 4.2. The system (4.1) is unstable if the characteristic roots of the matrix A contain at least one root with a positive real part.

Proof. The system (4.1) cannot be stable since when the function $\omega(t)$ is bounded, the stability in the sense of the Definition 1.1. would imply the Liapunov stability, while under the condition of the theorem the system (4.1) is Liapunov-unstable (see e.g. [5]).

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